ACTIONS OF COUNTABLE DISCRETE GROUPS ON THE FERMIONIC FACTOR

SAITO, Kazuyuki

数理解析研究所講究録
858巻 1994年 47-56

ACTIONS OF COUNTABLE DISCRETE GROUPS ON THE FERMIONIC FACTOR

東北大 理 斎藤和之(Kazuyuki SAITÔ)

The main theorem of Sullivan-Weiss-Wright shows that when A is the Dixmier algebra, if G_1 and G_2 are arbitrary countably infinite discrete groups and if α^{G_1}, α^{G_2} are, respectively, outer, ergodic actions on A as $*$-automorphism groups, then, (A,G_1,α^{G_1}) is <u>always</u> weakly equivalent to (A,G_2,α^{G_2}) (see below Definition 2), where the Dixmier algebra is the C*-algebra of all bounded complex Borel functions on the unit interval $[0,1]$, modulo the ideal of functions with meagre supports (This is the regular completion of the C*-algebra $C[0,1]$. See [1]).

The idea of the proof of this result is to construct an action of the dyadic group $\bigoplus Z_2$ directly, by gluing together parts of homeomorphisms in G_1 and, at the same time, split the base space into dyadic pieces which can be mapped into the Cantor set in a natural way (see [11]). This means that the classifications modulo meagre sets are very rough. So it seems reasonable to conjecture that an analogous result is true when $A = \hat{F}$, the regular completion of the Fermion algebra F.

In this report, we would like to give you a negative answer to this conjecture by a concrete construction of an outer, ergodic action β of each countably infinite discrete group G on $\hat{F}$ so that β^{G_1} is weakly equivalent to β^{G_2} if, and only if, G_1 is isomorphic to G_2.

This is a joint work with John D.M. Wright [8], [9] and [10].

We recall that the regular completion $\hat{B}$ of a separable unital C*-algebra B is the <u>unique</u> monotone complete C*-algebra in which B is canonically embedded in such a way that

(1) $\hat{B}$ is monotone generated by B,

(2) for each self-adjoint element b in $\hat{B}$, the set

$$(-\infty, b] \cap B_h = \{x \in B_h \mid x \leq b\}$$

has b as its supremum in $\hat{B}_h$, where C_h is the set of all self-adjoint elements of a *-algebra C.

In fact, let $\mathcal{B}_B$ be the Borel envelope of B and let $\mathcal{M}_B$ be the set of elements in $\mathcal{B}_B$ such that

$$\{\phi \in \partial_e X_B \mid m(\phi) \neq 0\}$$

is a meagre subset of $\partial_e X_B$, where $\partial_e X_B$ is the Baire space of all pure states of B relative to the weak*-topology. Then $\mathcal{B}_B/\mathcal{M}_B$ meets all the requirements of the properties of $\hat{B}$. When B is simple and infinite dimensional, then $\hat{B}$ is a wild AW*-factor of Type III. In particular, $\hat{F}$ is a monotone complete, wild factor of Type III. (See [14] and [15]. See also [2]). We call this $\hat{F}$, _temporarily_, _a_ _Fermionic_ _factor_.

The Fermion algebra F may be identified with $\bigotimes M_2(\mathbb{C})$, the infinite tensor product of a countably infinite family of the algebras $M_2(\mathbb{C})$ of all two by two matrices over the complex numbers field $\mathbb{C}$. ([4] and [12]).

Let G be any countably _infinite_ group. We are able to construct an action β^G of G on F called _the Bernoulli shift action_. We can show that each β_h (h is in G with $h \neq e$, where e is the neutral element of G) is an outer automorphism of F. (See [8]). The following lemma plays a key role in our discussions :

Lemma 1. Let θ be any *-automorphism of F. Then, there exists a unique extension of θ to a *-automorphism $\hat{\theta}$ of $\hat{F}$. If θ is an outer automorphism of F, then $\hat{\theta}$ is an outer automorphism of $\hat{F}$.

Since F is simple, we can apply Theorem 8 in [1]. So, using Lemma 1, we can extend our Bernoulli shift action β of G on F to an outer action of G, $\hat{\beta}$ on $\hat{F}$. In fact, if g and h are in G, then for each x in F, we have

$$\hat{\beta}_g(\hat{\beta}_h(x)) = \hat{\beta}_g(\beta_h(x)) = \beta_g(\beta_h(x)) = \beta_{gh}(x) = \hat{\beta}_{gh}(x).$$

The uniqueness of the extension of β_{gh} to $\hat{\beta}_{gh}$, now tells us that $\hat{\beta}_{gh} = \hat{\beta}_g\hat{\beta}_h$ for each pair g and h in G. Moreover, $\hat{\beta}_g$ $(g \neq e)$ is outer on $\hat{F}$. We call this action <u>the Bernoulli shift action</u> of G on $\hat{F}$.

Definition 1. Let M be a monotone complete C*-algebra. Let $g \to \gamma_g$ be an action of G on M. This action is said to be <u>ergodic</u> if the only projections in M which are invariant under this action are 0 and 1.

It is natural to ask: When is the Bernoulli shift action on $\hat{F}$ ergodic ?

Theorem 1. $\hat{\beta}$ is ergodic on $\hat{F}$.

<u>An outline of a proof</u>. We shall assume that $\hat{\beta}$ is not ergodic, that is, there exists a non-zero projection p in $\hat{F}$ such that $p \neq 1$ and $\hat{\beta}_g(p) = p$ for all g in G. Since $p \neq 0$, 1/2 is not

is not an upper bound for $(-\infty, p] \cap (F_0)_h$ (where F_0 is the canonical locally finite dimensional *-subalgebra of F). So, there is an a in $(-\infty, p] \cap (F_0)_h$ such that $a \not\leqq 1/2$. Thus we have $(2\|a^+\|)^{-1} < 1$ (where a^+ is the positive portion of a). Since $1 - p \neq 0$, there exists a b in $(-\infty, 1 - p] \cap (F_0)_h$ such that $\|b^+\| > (2\|a^+\|)^{-1}$. Because our action is asymptotically abelian, there exists g in G such that $\hat{\beta}_g(a)$, $\hat{\beta}_g(a^+)$, b, b^+ all commute with each other and

$$\|\hat{\beta}_g(a^+)b^+\| \quad \|\hat{\beta}_g(a^+)\|\cdot\|b^+\| = \|a^+\|\cdot\|b^+\|.$$

Moreover, since p is invariant under $\hat{\beta}$, we can show that

$$0 \leqq 2\hat{\beta}_g(a^+)b^+ \leqq 1.$$

So $\|a^+\|\cdot\|b^+\| = \|\hat{\beta}_g(a^+)b^+\| \leqq 1/2$. This is a contradiction. So, this action $\hat{\beta}$ of G on $\hat{F}$ is ergodic.

Next we shall discuss about weak equivalences of the Bernoulli shift actions. Before going into discussions, we shall prepare monotone complete cross products of C*-dynamical systems. Let (A, G, α) be a discrete C*-dynamical system over a monotone complete C*-algebra A, where G is a countably infinite discrete group. Let $A \bar{\otimes} \mathcal{L}(l^2(G))$ be the monotone complete tensor product of A by the Type I factor $\mathcal{L}(l^2(G))$ (see [3] for details. See also [7]). We shall explain briefly about this construction. The elements of the monotone complete tensor product $A \bar{\otimes} \mathcal{L}(l^2(G))$ may be identified with those matrices $(a_{g,h})$ which correspond to elements in $A'' \bar{\otimes} \mathcal{L}(l^2(G))$ and whose entries are in A, where A'' is the second dual of A and $A'' \bar{\otimes} \mathcal{L}(l^2(G))$ is the usual von Neumann tensor product. The definition of "addition", "scalar multiplication" and "involution" of those matrices is straightforward and correspond to these operations in $A''\bar{\otimes}\mathcal{L}(l^2(G))$. "Multiplication" is more subtle, because, when multiplication $(a_{g,h})$ by $(b_{g,h})$, we need to be able to assign a value in A to infinite sums such as

$$" \sum_k a_{g,k} b_{k,h} ".$$

The notion of the order convergence in A by Kadison and Pedersen is used to define

$$o\text{-}\sum_k a_{g,k} b_{k,h}$$

in such a way that $A\bar{\otimes}\mathcal{L}(l^2(G))$ is a monotone complete C*-algebra. Hamana ([3], see also [5] and [13]) defines the monotone complete cross product associated with $(A,G\ \alpha)$ to be the subalgebra of $A\bar{\otimes}\mathcal{L}(l^2(G))$ corresponding to the algebra of all matrices such that

$$a_{hg,kg} = \alpha_{g^{-1}}(a_{h,k})$$

for each g, h and k in G. We write $M(A,G,\alpha)$ to denote this algebra. We know that $M(A,G,\alpha)$ is a monotone closed *-subalgebra of $A\bar{\otimes}\mathcal{L}(l^2(G))$. Let

$$\pi : A \to M(A,G,\alpha)$$

be defined by

$$\pi(a) = (\delta_{h,k}\alpha_{h^{-1}}(a))$$

for any a in A. Then π is a *-isomorphism from A onto $D\cap M(A,G,\alpha)$, where D is the diagonal subalgebra of $A\bar{\otimes}\mathcal{L}(l^2(G))$, corresponding to those matrices which vanish off the diagonal. Let

$$E^{\#} : A\bar{\otimes}\mathcal{L}(l^2(G)) \to D$$

be defined by $E^{\#}((a_{g,h})) = (\delta_{g,h}a_{g,h})$. It is straightforward to verify that $E = E^{\#}|M(A,G,\alpha)$ is a <u>normal</u> conditional expectation from $M(A,G,\alpha)$ onto $\pi(A)$. For each g in G, let

$$U_g = (\delta_{h,gk})_{h,k \text{ in } G}.$$

Then U_g is a unitary in $M(A,G,\alpha)$ such that

(1) $U_g\pi(a)U_g{*} = \pi(\alpha_g(a))$ for all a in A and g in G.

Straightforward calculations show that $g \to U_g$ is a group isomorphism of G into the unitary group $U(M(A,G,\alpha))$ such that

(2) $E(U_g x U_g{*}) = U_g E(x) U_g{*} = \alpha_g(E(x))$ for all a in A and g in G.

Moreover, it follows that, for any x in $M(A,G,\alpha)$,

(3) $E(xU_g) = 0$ for each g in G, then $x = 0$,

and

(4) $E(U_g) = 0$ if $g \neq e$.

Definition 2. Let (A,G_1,α_1) and (A,G_2,α_2) be C*-dynamical systems over the same monotone complete C*-algebra A. If, there exists an isomorphism θ from $M(A,G_1,\alpha_1)$ onto $M(A,G_2,\alpha_2)$ such that θ maps $\pi_1(A)$ onto $\pi_2(A)$, then, (A,G_1,α_1) and (A,G_2,α_2) are said to be weakly equivalent. (See [7]).

The main purpose of the rest of this report is to outline a proof of the following:

Theorem 2. Let G_1 and G_2 be discrete countably infinite groups. For each j, let $\hat{\beta}^j$ be the Bernoulli shift action of G_j on $\hat{F}$. Let $(\hat{F},G_1,\hat{\beta}^1)$ and $(\hat{F},G_2,\hat{\beta}^2)$ be weakly equivalent. Then G_1 is isomorphic to G_2.

If A is a monotone complete factor and α is an outer action of G on A. Then, $M(A,G,\alpha)$ is a factor. In particular, $M(\hat{F},G,\hat{\beta})$ is a wild, monotone complete factor of Type III. ([3], see also [5] and [13]).

Let us simplify the notations. Let B be any unital C*-algebra and let $U(B)$ be the group of all unitaries in B. Let A be a unital *-subalgebra of B. We define an A-normalizing unitary to be any u in $U(B)$ such that $uAu^* = A$. We denote the group of all A-normalizing unitaries in B by $N(B:A)$. Let G be a group. We define a <u>G system</u> to be an ordered quadruple (B,A,U_G,E) where E is a conditional expectation from B onto A and U_G is an isomorphic image of G in $N(B:A)$ such that the following conditions are satisfied:

(a) Let u be in U_G. If the automorphism $\mathrm{Ad}u$ is also implemented by some unitary in A, then $u = 1$.

(b) For each u in U_G and each b in B,

$$E(ubu^*) = uE(b)u^*.$$

(c) Let b be in B. If $E(ubv) = 0$ for each pair u and v in U_G, then $b = 0$.

(d) Whenever u in U_G satisfies that $u \neq 1$, then $E(u) = 0$.

It is easy to check that $(M(\hat{F},G,\hat{\beta}),\pi(\hat{F}),U_G,E)$ satisfies the above properties because of (1), (2), (3) and (4). Moreover, we can show the following:

Lemma 2. Let (B,A,U_G,E) be any G system and let the centre Z_A of A be one dimensional. Let x be in $B \cap A'$. Then x is a scalar multiple of 1.

Using this lemma and analysing carefully the structure of N(B:A), we have the following:

Theorem 3. Let G_1 and G_2 be discrete groups. Let (B_1,A_1,U_{G_1},E_1) and (B_2,A_2,U_{G_2},E_2) be, respectively, a G_1 system and a G_2 system. Let θ be a *-isomorphism of B_2 onto B_1 such that θ maps A_2 onto A_1. Let A_1 have one dimensional centre. Then U_{G_1} and U_{G_2} are isomorphic.

Theorem 2 is an easy consequence of Theorem 3.

References

[1] J. Dixmier, Sur certains espaces consideres par M.H. Stone, Summa Math. Brasil., 11(1951), 151-182.

[2] G.A. Elliott, K. Saitô and J.D.M. Wright, Embedding AW*-algebras as double commutants in Type I algebras, J. London Math. Soc., 28(1983), 376-384.

[3] M. Hamana, Tensor products for monotone complete C*-algebras, I, II, Japan. J. Math., 8(1982), 261-295.

[4] R.V. Kadison and J.R. Ringrose, Fundamentals of the theory of operator algebras, vol. II, Academic Press(1986).

[5] K. Saitô, AW*-algebras with monotone convergence property and examples by Takenouchi and Dyer, Tôhoku Math. J., 31(1979), 31-40.

[6] K. Saitô and J.D.M. Wright, Outer automorphisms of regular completions, J. London Math. Soc., 27(1983), 150-156.

[7] K. Saitô and J.D.M. Wright, On tensor products of monotone complete algebras, Quart. J. Math. Oxford, 35(1984), 209-221.

[8] K. Saitô, Dynamic actions on monotone complete factors, Quart. J. Math. Oxford, 44(1993), 239-247.

[9] K. Saitô and J.D.M. Wright, A continuum of discrete group actions on a Fermionic factor, To appear in Quart. J. Math. Oxford, 1993.

[10] K. Saitô and J.D.M. Wright, Ergodic actions on the Fermionic factor, To appear in Quart. J. Math. Oxford, 1993.

[11] D. Sullivan, B. Weiss and J.D.M. Wright, Generic dynamics and monotone complete C*-algebras, Trans. Amer. Math. Soc., 395(1986), 795-809.

[12] Z. Takeda, Inductive limit and infinite direct product of operator algebras, Tôhoku Math. J., 7(1955), 67-86.

[13] O. Takenouchi, A non W*, AW*-factor, C*-algebras and applications to Physics (Proc. Second Japan-U.S.A. Seminar, Los Angeles (1977)), Lecture Notes in Math., 650(1978), 135-139.

[14] J.D.M. Wright, Regular σ-completion of C*-algebras, J. London Math. Soc., 12(1976), 299-309.

[15] J.D.M. Wright, Wild AW*-factors and Kaplansky-Rickart C*-algebras, J. London Math. Soc., 13(1976), 83-89.

Mathematical Institute
Tôhoku University
Sendai, Japan